

e. None of the above.

Section 8.2: Nonhomogeneous linear systems

The story so far:

We have solved homogeneous systems x' = Ax when

• A is diagonalizable.

The approach uses eigenvalues and eigenvectors to give a fundamental matrix.

Exponential give a way to write this., in that we can compute e^A from the fundamental matrix.

• A is not diagonalizable , in some special cases. We did this by computing e^A directly for nilpotent matrices. It is possible to take the method further to get complete generality, but we do not do this. Go to Math 4242.

In this section:

• we do non-homogeneous equations

 $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{t}).$

There are two approaches:

- the method of undetermined coefficients
- the method of variation of parameters

These extend the same methods that we applied in the case of equations in a single variable.

Method 1 is more elaborate when F(t) is a function involving the same exponential as a solution of the corresponding homogeneous equation.

Method 2 deals with this automatically. Both methods are computationally intense! Page 489 question 13 + extra Find a particular solution to $x' = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \times + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{2t}$ Substitute $\begin{bmatrix} 2t \\ 2b \\ -3 \end{bmatrix} = e^{2t} \begin{bmatrix} 2a+b+2 \\ a+2b-3 \end{bmatrix}$ Solution 1: method of undetermined coefficients. It is helpful (but not totally necessary for this

Solution 1: method of undetermined coefficients It is helpful (but not totally necessary for this problem) to solve the corresponding homogeneous equation:

Characteristic polynomial: $(2-\lambda)^2 = \lambda^2 + 4\lambda + 3$

 $= (\lambda - 1) (\lambda - 3)$

Eigenvalues and eigenvectors:

 $\lambda = [\text{ Null } [i, j] \ge e - \text{vector } [i]$ $\lambda = 3 \text{ Null } [i, j] \ge e - \text{vector } [i]$ Fundamental matrix: $\overline{p}(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$

We try a solution
$$x = \begin{bmatrix} a \\ b \end{bmatrix} e^{2t}$$

 $x'(t) = 2\begin{bmatrix} a \\ b \end{bmatrix} e^{2t}$

$$k(t) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t}$$

b = -2 a = 3.

a = 2a+b } give a+b=3(a+b), so a+b=0b = a+2b } b = -aFind a particular solution to $x' = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{t}$ Page 489 question 13 + extra atc = 2c + dt 27 a = c + d + 2 b + d = c + 2d - 35 gtve = b = c + d - 3Solution 1: method of undetermined coefficients So C+d = a - 2 = b + 3 = -a + 3, 2a = 5a = 5/2, b = -5/2 c+d = 12Also, these linear equations could have been so that there is a solution to the homogeneous equation with the same power of e: $x(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ solved more formally with row reduction Eventually $q = \frac{5}{2}$ $b = \frac{-5}{2}$, $c + d = \frac{1}{2}$ There are infinitely many solutions e.g. $c = \frac{1}{2}$, d = 0 or c = 0, $d = \frac{1}{2}$ $N = \frac{1}{4} d = \frac{1}{4}$. Solution Try X(t)=[]] tet + [] et Find x', Substrike, get equation trab, c.t. Notice: trying $x(t) = \begin{bmatrix} e \\ b \end{bmatrix} e^t$ doesn't work: The details of this where written in after class ; $x'(t) = \begin{bmatrix} e \\ d \end{bmatrix} e^t = \begin{pmatrix} 2c+d \\ e+2d \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^t$ so $k(t) = \begin{bmatrix} 9 \\ 4 \end{bmatrix} tet + \begin{bmatrix} 6 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} et'$ $= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \times + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{t} = \begin{bmatrix} 2a+6 \\ a+26 \end{bmatrix} te^{t} + \begin{bmatrix} 2c+d+2 \\ c+2d-3 \end{bmatrix} e^{t}$ $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d+2 \\ c+2d-3 \end{bmatrix}, \ c+d = -2 = 3.$ Equate coefficients of tet and of et There is no solution.

In the last calculation we eventually got

solutions such as

$$X(t) = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix} tet + \begin{bmatrix} 1 \\ 2 \\ -5/2 \end{bmatrix} et$$

When we tried a solution could a spear here

 $\begin{bmatrix} a \\ b \end{bmatrix} t e^t + \begin{bmatrix} c \\ d \end{bmatrix} e^t$

it was necessary to include the term $\begin{bmatrix} a \\ d \end{bmatrix} e^{t}$ unlike the case of single variable method of undetermined coefficients, because only some of the functions of the form $\begin{bmatrix} c \\ d \end{bmatrix} e^{t}$

are solutions of the homogeneous equation, namely those that are scalar multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

a. $\begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $\begin{bmatrix} e^t \\ e^t \end{bmatrix}$ Pre-class Warm-up!!! When A = [] we can calculate that b. $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $t \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ $e^{At} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$ (c. [et] and [tet] Docs the help is solve the system of equations x' = Ax? d. $X = \begin{bmatrix} e^t & te^t \\ o & e^t \end{bmatrix}$ Which of the following is a basis for the e. Not enough information to say Solution space? We know: $e^{\pm} = \overline{\Phi} \overline{\Phi}(0)^{-1}$ where the columns of $\overline{\Phi}$ are a beisis pr the column space. The columns of $\overline{\Phi} \overline{\Phi}(0)^{-1}$ are linear comprised cols of 2. They are independent.

Method of variation of parameters

We solve x' = Ax + F(t), x(0) = c. The homogeneous equation has solution $x = e^{At} = \Phi \Phi(0)^{1}$

Try a solution $x = e^{At}c$ where c = c(t)is a function of t. Then $x' = Ae^{At}c + e^{At}c' = AxtF$

 $= Ae^{At}c + F$ Thus $e^{At}c' = F$, $c' = e^{-At}F$

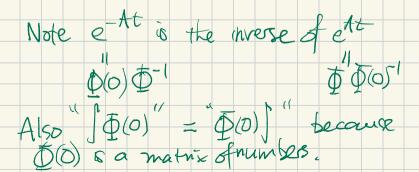
 $C = \int e^{-At} F dt$ Me solution is $x = e^{At}c = e^{At}\int e^{-At} F dt$

 $= \overline{\Phi} \overline{\Phi} \overline{0} \overline{0} \overline{0} \overline{\Phi} \overline{0} \overline{\Phi} \overline{0} \overline{\Phi}^{-1} F dt$

 $= \Phi \int \Phi^{-1} F at$

The last is Theorem 1, formula (22), on page 486.

$$x = \Psi \int \Phi^{-1} F dt$$



Page 489 question 13 + extra Find a particular solution to $x' = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{2t}$ $x = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2e^{4t} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 2e^{2t} \\ -3e^{2t} \end{bmatrix} dt$

Solution 2: method of variation of parameters. We solve the corresponding homogeneous equation:

Characteristic polynomial: $(\lambda - 3)(\lambda - 1)$

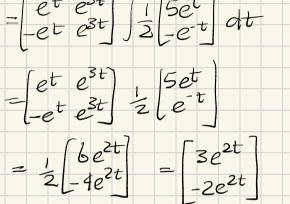
Eigenvalues and eigenvectors:

 $\lambda = | e - vec \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda = 3 e - vec = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Fundamental matrix:

Recall the formula: $\chi = \vec{\phi} \left(\vec{\Phi}^{-1} F(t) dt \right)$

$$D^{-1} = \frac{1}{e^{t}e^{3t} + e^{t}e^{3t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix}$$
$$= \frac{1}{2e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix}$$



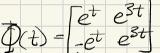
which is the same as the answer we got before.

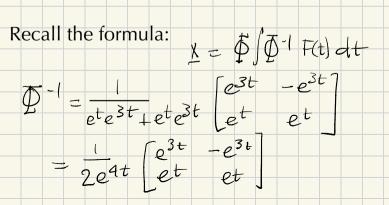
Page 489 question 13 + extra Find a particular solution to $x' = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} X + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{t}$

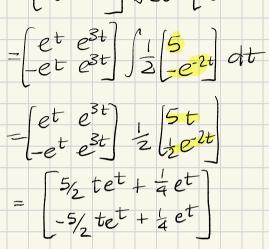
This is the same as before with et, not er

Solution 2: method of variation of parameters. We solve the corresponding homogeneous equation:

Fundamental matrix:







x =

which is the same as the answer we got before.

with the choice
$$C = d = \frac{1}{4}$$

Question: Which did you like better?

a. the method of undetermined coefficients

b. the method of variation of parameters