Question: What is e^{At} where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$?

a. $\begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$

b. $\begin{bmatrix} e^t & t \\ 0 & e^t \end{bmatrix}$

c. $\begin{bmatrix} te^t & e^t \\ 0 & te^t \end{bmatrix}$

d. $\begin{bmatrix} e^t & \frac{1}{2}t^2 \\ 0 & e^t \end{bmatrix}$

e. None of the above.

Section 8.2: Nonhomogeneous linear systems

The story so far:
We have solved homogeneous systems $x' = Ax$ when

- A is diagonalizable.
  The approach uses eigenvalues and eigenvectors to give a fundamental matrix.
  Exponential give a way to write this., in that we can compute e^A from the fundamental matrix.

- A is not diagonalizable , in some special cases. We did this by computing e^A directly for nilpotent matrices. It is possible to take the method further to get complete generality, but we do not do this. Go to Math 4242.

In this section:

- we do non-homogeneous equations $x' = Ax + F(t)$.

There are two approaches:

- the method of undetermined coefficients
- the method of variation of parameters

These extend the same methods that we applied in the case of equations in a single variable.

Method 1 is more elaborate when $F(t)$ is a function involving the same exponential as a solution of the corresponding homogeneous equation.

Method 2 deals with this automatically.
Both methods are computationally intense!

Page 489 question 13 + extra

Find a particular solution to $x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{2t}$

Solution 1: method of undetermined coefficients. It is helpful (but not totally necessary for this problem) to solve the corresponding homogeneous equation:

Characteristic polynomial: $(2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$

$= (\lambda - 1)(\lambda - 3)$

Eigenvalues and eigenvectors:

$\lambda = 1$ Null $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$: e-vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 3$ Null $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ e-vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$

We try a solution $x = \begin{bmatrix} a \\ b \end{bmatrix} e^{2t}$

$x'(t) = 2\begin{bmatrix} a \\ b \end{bmatrix} e^{2t}$

Substitute : $e^{2t}\begin{bmatrix} 2a \\ 2b \end{bmatrix} = e^{2t}\begin{bmatrix} 2a+b+2 \\ a+2b-3 \end{bmatrix}$

Equations $2a = 2a+b+2$

$2b = a+2b-3$

$b = -2 \quad a = 3$.

A particular solution is

$x(t) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t}$.

Page 489 question 13 + extra

Find a particular solution to $x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^t$

Last time the e^t was e^{2t}.

Solution 1: method of undetermined coefficients.

The fundamental matrix was:

$$\Phi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$$

so that there is a solution to the homogeneous equation with the same power of e:

$$x(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

Solution Try $x(t) = \begin{bmatrix} a \\ b \end{bmatrix} te^t + \begin{bmatrix} c \\ d \end{bmatrix} e^t$

Find x', Substitute, get equations for a,b,c,d.

The details of this were written in after class:

$$x'(t) = \begin{bmatrix} a \\ b \end{bmatrix} te^t + \left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) e^t$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^t = \begin{bmatrix} 2a+b \\ a+2b \end{bmatrix} te^t + \begin{bmatrix} 2c+d+2 \\ c+2d-3 \end{bmatrix} e^t$$

Equate coefficients of te^t and of e^t:

$\left.\begin{aligned} a &= 2a+b \\ b &= a+2b \end{aligned}\right\}$ give $a+b = 3(a+b)$, so $a+b=0$, $b=-a$.

$\left.\begin{aligned} a+c &= 2c+d+2 \\ b+d &= c+2d-3 \end{aligned}\right\}$ give $\begin{aligned} a &= c+d+2 \\ b &= c+d-3 \end{aligned}$

so $c+d = a-2 = b+3 = -a+3$, $2a=5$

$a = 5/2$, $b = -5/2$ $c+d = \frac{1}{2}$

Also, these linear equations could have been solved more formally with row reduction.

Eventually $a = \frac{5}{2}$ $b = \frac{-5}{2}$, $c+d = \frac{1}{2}$

There are infinitely many solutions

e.g. $c = \frac{1}{2}, d = 0$ or $c=0, d=\frac{1}{2}$ or $c=\frac{1}{4}, d=\frac{1}{4}$

See next page!

Notice: trying $x(t) = \begin{bmatrix} c \\ d \end{bmatrix} e^t$ doesn't work:

$$x'(t) = \begin{bmatrix} c \\ d \end{bmatrix} e^t = \left(\begin{bmatrix} 2c+d \\ c+2d \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) e^t$$ so

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d+2 \\ c+2d-3 \end{bmatrix}, \quad c+d = -2 = 3.$$

There is no solution.

In the last calculation we eventually got solutions such as

$$x(t) = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix} te^t + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t$$

Other vectors such as $\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$ could appear here

When we tried a solution

$$\begin{bmatrix} a \\ b \end{bmatrix} te^t + \begin{bmatrix} c \\ d \end{bmatrix} e^t$$

it was necessary to include the term $\begin{bmatrix} c \\ d \end{bmatrix} e^t$ unlike the case of single variable method of undetermined coefficients, because only some of the functions of the form $\begin{bmatrix} c \\ d \end{bmatrix} e^t$

are solutions of the homogeneous equation, namely those that are scalar multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

Pre-class Warm-up!!!

When $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we can calculate ~~that~~

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

Does this help us solve the system of equations $x' = Ax$?

Which of the following is a basis for the solution space?

a. $\begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $t\begin{bmatrix} e^t \\ e^t \end{bmatrix}$

b. $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $t\begin{bmatrix} e^t \\ e^t \end{bmatrix}$

√ c. $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} te^t \\ e^t \end{bmatrix}$

d. $x = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$

e. Not enough information to say.

We know: $e^{At} = \Phi\, \Phi(0)^{-1}$ where the columns of Φ are a basis for the solution space. The columns of $\Phi\, \Phi(0)^{-1}$ are linear combns of cols of Φ. They are independent.

Method of variation of parameters

We solve $x' = Ax + F(t)$, $x(0) = c$.
The homogeneous equation has solution

$$x = e^{At}\underline{c} \qquad e^{At} = \Phi\,\Phi(0)^{-1}$$

Try a solution $x = e^{At}\underline{c}$ where $\underline{c} = \underline{c}(t)$ is a function of t.

Then

$$x' = Ae^{At}\underline{c} + e^{At}\underline{c}' = Ax + F$$

$$= Ae^{At}\underline{c} + F$$

Thus $e^{At}\underline{c}' = F$, $\underline{c}' = e^{-At}F$

$$c = \int e^{-At} F\,dt$$

The solution is $x = e^{At}c = e^{At}\int e^{-At}F\,dt$

$$= \Phi\,\Phi(0)^{-1}\int \Phi(0)\Phi^{-1}F\,dt$$

$$= \Phi\int \Phi^{-1}F\,dt$$

The last is Theorem 1, formula (22), on page 486.

$$x = \Phi\int\Phi^{-1}F\,dt$$

Note e^{-At} is the inverse of e^{At}

$$e^{-At} = \Phi(0)\Phi^{-1} \qquad e^{At} = \Phi\,\Phi(0)^{-1}$$

Also "$\int\Phi(0)$" = "$\Phi(0)\int$" because $\Phi(0)$ is a matrix of numbers.

Page 489 question 13 + extra

Find a particular solution to $x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{2t}$

Solution 2: method of variation of parameters.
We solve the corresponding homogeneous equation:

Characteristic polynomial: $(\lambda-3)(\lambda-1)$

Eigenvalues and eigenvectors:

$\lambda = 1 \quad \text{e-vec} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda = 3 \quad \text{e-vec} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Fundamental matrix:

$$\Phi(t) = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix}$$

Recall the formula:

$$\underline{x} = \Phi \int \Phi^{-1} F(t)\, dt$$

$$\Phi^{-1} = \frac{1}{e^{t}e^{3t} + e^{t}e^{3t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix}$$

$$= \frac{1}{2e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix}$$

$$x = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix} \int \frac{1}{2e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 2e^{2t} \\ -3e^{2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix} \int \frac{1}{2} \begin{bmatrix} 5e^{t} \\ -e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 5e^{t} \\ e^{-t} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 6e^{2t} \\ -4e^{2t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} \\ -2e^{2t} \end{bmatrix}$$

which is the same as the answer we got before.

Page 489 question 13 + extra

Find a particular solution to $x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{t}$

This is the same as before with e^t, not e^{2t}

Solution 2: method of variation of parameters.

We solve the corresponding homogeneous equation:

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$

Recall the formula:

$$\underline{x} = \Phi \int \Phi^{-1} F(t)\, dt$$

$$\Phi^{-1} = \frac{1}{e^t e^{3t} + e^t e^{3t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix}$$

$$= \frac{1}{2e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix}$$

$$x = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \int \frac{1}{2e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix} \begin{bmatrix} 2e^{t} \\ -3e^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \int \frac{1}{2} \begin{bmatrix} 5 \\ -e^{-2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 5t \\ \frac{1}{2} e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} te^t + \frac{1}{4} e^t \\ -\frac{5}{2} te^t + \frac{1}{4} e^t \end{bmatrix}$$

which is the same as the answer we got before.

with the choice $c = d = 1/4$

Question:
Which did you like better?

a. the method of undetermined coefficients

b. the method of variation of parameters